

Fast robust SUR with economical and actuarial applications

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Abstract

The seemingly unrelated regression (SUR) model is a generalization of a linear regression model consisting of more than one equation, where the error terms of these equations are contemporaneously correlated. The standard Feasible Generalized Linear Squares (FGLS) estimator is efficient as it takes into account the covariance structure of the errors, but it is also very sensitive to outliers. The robust SUR estimator of Bilodeau and Duchesne (Canadian Journal of Statistics, 28:277–288, 2000) can accommodate outliers, but it is hard to compute. First we propose a fast algorithm, FastSUR, for its computation and show its good performance in a simulation study. We then provide diagnostics for outlier detection and illustrate them on a real data set from economics. Next we apply our FastSUR algorithm in the framework of stochastic loss reserving for general insurance. We focus on the General Multivariate Chain Ladder (GMCL) model that employs SUR to estimate its parameters. Consequently, this multivariate stochastic reserving method takes into account the contemporaneous correlations among run-off triangles and allows structural connections between these triangles. We plug in our FastSUR algorithm into the GMCL model to obtain a robust version.

Keywords: Seemingly Unrelated Regression; Feasible Generalized Least Squares; S-estimator; Outlier; Claims Reserving.

1 Introduction

Many studies in econometrics, insurance and finance are based on regression models containing more than one equation. Unconsidered factors that influence the error term in one equation often also influence the error terms in other equations. Ignoring this dependence structure of the error terms and estimating these equations separately using ordinary least squares (OLS) leads to inefficient estimates. Therefore, in [1] the Seemingly Unrelated Regressions (SUR) model is proposed. It is composed of several regression equations that are linked by the fact that their error terms are contemporaneously correlated. This system of structurally related equations is simultaneously estimated with a generalized least squares (GLS) estimator that takes the covariance structure of the error terms into account. An extensive summary of the literature dealing with the SUR model and its various extensions can be found in [2] and [3].

The SUR models have found considerable use in many applications in econometrics, finance and insurance. For example, in [4], [5] and [6] the effects of financial crises and shocks on the insurance industry, banks and stock markets is studied whereas in [7] and [8] relationships are examined between income and consumption, inflation and stock markets, ethnic diversity and economic growth. In this paper we illustrate the SUR model on studying the relation between the Foreign Direct Investment (FDI) by multinational corporations and several macroeconomic variables. Data are available for six countries over the period 1981-2012. This yields $m = 6$ regression blocks, each with $n = 32$ observations. Since there is a possible existence of common factors (such as economy-wide or worldwide shocks) that influence all countries at the same time, cross-correlations between the error terms may exist. This motivates the use of the SUR model for fitting this system of equations. As another application we consider claims reserving in general insurance, which is a major actuarial issue. Recently the multivariate reserving approach has received extensive attention, see e.g. [9, 10, 11, 12, 13]. We will focus on the general multivariate chain ladder model of Zhang [14], where the parameters are estimated using SUR.

Since the common GLS estimation procedure for the SUR model is based on the classical covariance matrix and OLS estimation, the method as a whole is very sensitive to outliers. Outliers are observations that differ from the majority of the data and it is well known that they can have a large impact on classical statistical methods. Therefore, robust alternatives have already been presented in the literature. In [15] a robust version of the SUR model is proposed based on M-estimators. Since this procedure is not affine equivariant and does not take full account of the multivariate nature of the problem, a

method based on S-estimators is introduced in [16]. S-estimators were introduced for regression in [17], whereas in [18] and [19] S-estimators for multivariate location and scatter are studied. The robust SUR method in [16] combines both types of S-estimators, which results in an estimator that is regression and affine equivariant and is able to detect multivariate outliers.

First we resume the SUR model and the standard GLS estimators in Section 2. Section 3 describes the robust SUR method in [16] and proposes a new algorithm for its computation. We then show its good performance in a simulation study (Section 4). In Section 5 we provide a diagnostic tool to detect outlying observations, and illustrate it on a data set from macroeconomics. Section 6 elaborates on the use of the robust SUR method in the context of stochastic loss reserving, whereas Section 7 provides some directions for further research.

2 Classical SUR

In general the SUR model comprises m linear equations (also called blocks), each of which is assumed to satisfy the Gauss-Markov conditions. Each block contains an equal number n of observations, hence the system can be written as

$$\begin{cases} y_1 &= \mathbf{X}_1\beta_1 + \varepsilon_1 \\ &\vdots \\ y_m &= \mathbf{X}_m\beta_m + \varepsilon_m \end{cases} \quad (1)$$

where $y_j = (y_{1j}, y_{2j}, \dots, y_{nj})'$ is the $n \times 1$ response vector of the j th block, \mathbf{X}_j is the $n \times p_j$ matrix of explanatory variables, β_j is the $p_j \times 1$ vector of regression parameters while ε_j corresponds to the $n \times 1$ error vector which satisfies $E(\varepsilon_j) = 0$ and $\text{Cov}(\varepsilon_j) = \sigma_{jj}\mathbf{I}_n$ (for all $j = 1, \dots, m$). Here, \mathbf{I}_n denotes the $n \times n$ identity matrix. Note that each block has its own dependent variable and potentially different sets of exogenous explanatory variables. We further assume for each j that $\text{rank}(\mathbf{X}_j) = p_j \leq n$ to avoid singular solutions. These seemingly unrelated regression blocks are linked through their zero mean error structure. It is namely assumed that the error vectors are contemporaneously but not serially correlated herein. This means that for given observations i and l , across the regression equations j and k , it holds that

$$\begin{aligned} E(\varepsilon_{ij}\varepsilon_{ik}) &= \sigma_{jk} \text{ for all } i = 1, 2, \dots, n \\ E(\varepsilon_{ij}\varepsilon_{lj}) &= 0 \quad \text{when } i \neq l \\ E(\varepsilon_{ij}\varepsilon_{lk}) &= 0 \quad \text{when } j \neq k, \quad i \neq l \end{aligned}$$

The system of m seemingly unrelated regression equations (1) can be stacked in two equivalent compact matrix forms. First, we can express it as a multiple linear regression model:

$$y = \mathbf{X}\beta + \varepsilon \quad (2)$$

where $y = (y'_1, \dots, y'_m)'$ is the $nm \times 1$ response vector,

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{X}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}_m \end{bmatrix}$$

is the $nm \times p$ structured design matrix, with $p = \sum_{j=1}^m p_j$, $\beta = (\beta'_1, \dots, \beta'_m)'$ is the $p \times 1$ parameter vector and $\varepsilon = (\varepsilon'_1, \dots, \varepsilon'_m)'$ is the error vector with

$$\text{Cov}(\varepsilon) = \mathbf{\Sigma} \otimes \mathbf{I}_n = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mm} \end{bmatrix} \otimes \mathbf{I}_n = \begin{bmatrix} \sigma_{11}\mathbf{I}_n & \sigma_{12}\mathbf{I}_n & \cdots & \sigma_{1m}\mathbf{I}_n \\ \sigma_{21}\mathbf{I}_n & \sigma_{22}\mathbf{I}_n & \cdots & \sigma_{2m}\mathbf{I}_n \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1}\mathbf{I}_n & \sigma_{m2}\mathbf{I}_n & \cdots & \sigma_{mm}\mathbf{I}_n \end{bmatrix}.$$

Note that σ_{jj} is the variance of the error term in the j th equation, whereas σ_{jk} is the covariance between the error terms in equation j and the error terms in equation k .

Another formulation of the SUR model uses the multivariate linear regression model:

$$\mathbf{Y} = \tilde{\mathbf{X}}\mathbf{B} + \mathcal{E} \quad (3)$$

where $\mathbf{Y} = (y_1, \dots, y_m)$ is the $n \times m$ response matrix, $\tilde{\mathbf{X}} = [\mathbf{X}_1, \dots, \mathbf{X}_m]$ the $n \times p$ design matrix,

$$\mathbf{B} = \begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_m \end{bmatrix} = \text{diag}(\beta_1, \beta_2, \dots, \beta_m) \quad (4)$$

the structured $p \times m$ parameter matrix and $\mathcal{E} = (\varepsilon_1, \dots, \varepsilon_m)$ the error matrix with $\text{Cov}(\mathcal{E}) = \mathbf{I}_n \otimes \mathbf{\Sigma}$. Equivalently we can write the error matrix as $\mathcal{E} = \mathbf{Y} - \tilde{\mathbf{X}}\mathbf{B} = (e_1, \dots, e_n)'$ with e_i the m -dimensional vector containing the errors of the i th observation in each block.

Each equation in (1) could be estimated separately using the OLS estimator but this would ignore the covariance structure of the errors. A more efficient estimator is obtained as the GLS estimator with weight matrix $\mathbf{W} = \text{Cov}(\varepsilon)$. As $\mathbf{\Sigma}$ is typically

unknown, a Feasible GLS (FGLS) estimator is preferred that replaces the unknown \mathbf{W} with a consistent estimate. The FGLS estimator is an iterative two-step procedure that uses estimates for β to estimate Σ , which is then used to improve the regression estimates $\hat{\beta}$. To start, each equation is estimated by OLS, yielding $\hat{\beta}_j$. Then iteratively:

- The residuals $\hat{\varepsilon}_j = y_j - \mathbf{X}_j \hat{\beta}_j$ from the m equations are used to estimate the error covariance matrix $\hat{\mathbf{W}} = \hat{\Sigma} \otimes \mathbf{I}_n$ with $\hat{\Sigma} = \frac{1}{n}(\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_m)'(\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_m)$.
- New estimates of β are obtained as

$$\hat{\beta} = (\mathbf{X}'\hat{\mathbf{W}}^{-1}\mathbf{X})^{-1}\mathbf{X}'\hat{\mathbf{W}}^{-1}y = (\mathbf{X}'(\hat{\Sigma}^{-1} \otimes \mathbf{I}_n)\mathbf{X})^{-1}\mathbf{X}'(\hat{\Sigma}^{-1} \otimes \mathbf{I}_n)y.$$

The estimated covariance matrix of $\hat{\beta}$ is given by $\text{Cov}(\hat{\beta}) = (\mathbf{X}'\hat{\mathbf{W}}^{-1}\mathbf{X})^{-1}$.

3 Robust SUR

3.1 The robust SUR method

We first recall the definition of the S-estimator of the SUR model, as introduced in [16]: it is defined as the couple $(\hat{\mathbf{B}}, \hat{\mathbf{S}})$ which minimizes $|\mathbf{S}|$ under the condition

$$\frac{1}{n} \sum_{i=1}^n \rho \left(\sqrt{e_i(\mathbf{B})' \mathbf{S}^{-1} e_i(\mathbf{B})} \right) = b \quad (5)$$

over all (\mathbf{B}, \mathbf{S}) with $\mathbf{B} = \text{diag}(b_1, \dots, b_m) \in \mathbb{R}^{p \times m}$, $b_j \in \mathbb{R}^{p_j \times 1}$ for all $j = 1, \dots, m$, $e_i(\mathbf{B})'$ the i th row of $\mathbf{Y} - \tilde{\mathbf{X}}\mathbf{B}$, and \mathbf{S} an $m \times m$ symmetric positive definite (SPD) matrix. In order to obtain a robust estimator which is consistent and asymptotically normal, ρ should satisfy the following conditions:

(C1) ρ is symmetric around zero and twice continuously differentiable

(C2) $\rho(0) = 0$ and ρ is strictly increasing on $[0, c_0]$ and constant on $[c_0, \infty[$ for some $c_0 > 0$.

The constant b can be computed as $E_{F_0}[\rho(|\mathbf{e}|)]$ where $\mathbf{e} \sim F_0$. As such, $F_0 = N_m(\mathbf{0}, \mathbf{I}_m)$ ensures consistency at the model with normal errors. For ρ one often chooses the function

$$\rho(x) = \begin{cases} \frac{x^2}{2} - \frac{x^4}{2c^2} + \frac{x^6}{6c^4} & \text{for } |x| \leq c \\ \frac{c^2}{6} & \text{for } |x| > c \end{cases}$$

where c is an appropriate tuning constant [17]. The derivative of this function is known as Tukey's bisquare function:

$$\rho'(x) = \psi(x) = \begin{cases} x \left(1 - \left(\frac{x}{c}\right)^2\right)^2 & \text{for } |x| \leq c \\ 0 & \text{for } |x| > c. \end{cases}$$

We will use this ρ -function throughout the paper. For fixed b , the value of the tuning constant c determines the breakdown value, see [20]. For high-dimensional regressors, one could also consider other ρ -functions that downweight outliers more appropriately, see [21].

In addition to the minimization condition mentioned above the robust SUR estimators of β and Σ also satisfy the following equations [16]:

$$\beta = (\mathbf{X}'(\Sigma^{-1} \otimes \mathbf{D})\mathbf{X})^{-1} \mathbf{X}'(\Sigma^{-1} \otimes \mathbf{D})\mathbf{y} \quad (6)$$

$$\Sigma = m(\mathbf{Y} - \tilde{\mathbf{X}}\mathbf{B})' \mathbf{D}(\mathbf{Y} - \tilde{\mathbf{X}}\mathbf{B}) / \sum_{i=1}^n v(d_i) \quad (7)$$

where $d_i^2 = \mathbf{e}_i' \Sigma^{-1} \mathbf{e}_i$ and $\mathbf{D} = \text{diag}(w(d_i))$, for $w(u) = \rho'(u)/u$ and $v(u) = \rho'(u)u - \rho(u) + b$.

3.2 The FastSUR algorithm

S-estimators have good robustness properties, but they are computationally expensive. The original resampling algorithm in [17] was first improved by the SURREAL algorithm [22]. Next, a better performance was achieved by the FastS algorithm introduced in [23] for regression and in [24] for multivariate location and scatter. This is currently the most popular algorithm and is e.g. included in the R packages `robustbase` and `rrcov`, and the FSDA Matlab toolbox [25].

For the computation of the robust SUR method (5), Bilodeau and Duchesne [16] adapted the SURREAL algorithm of [22]. Here we propose the FastSUR algorithm, which implements the ideas of the FastS algorithm into the robust SUR estimator.

First, the \mathbf{S} in (5) is written as $\sigma^2 \mathbf{\Gamma}$ with $|\mathbf{\Gamma}| = 1$ and $\sigma = |\mathbf{S}|^{1/2m}$, so that the equivalent objective is to find the triplet $(\hat{\mathbf{B}}, \hat{\mathbf{\Gamma}}, \hat{\sigma})$ that minimizes σ under the restriction

$$\frac{1}{n} \sum_{i=1}^n \rho \left(\frac{\sqrt{\mathbf{e}_i(\mathbf{B})' \mathbf{\Gamma}^{-1} \mathbf{e}_i(\mathbf{B})}}{\sigma} \right) = b$$

over all $(\mathbf{B}, \mathbf{\Gamma}, \sigma)$ where $\mathbf{B} = \text{diag}(\beta_1, \dots, \beta_m) \in \mathbb{R}^{p \times m}$, $\mathbf{\Gamma}$ is an $m \times m$ SPD matrix with $|\mathbf{\Gamma}| = 1$ and σ is a positive scalar. The robust SUR estimates are then given by $(\hat{\mathbf{B}}, \hat{\Sigma} = \hat{\sigma}^2 \hat{\mathbf{\Gamma}})$.

The algorithm starts with N initial estimates $(\hat{\mathbf{B}}_1^{(0)}, \hat{\mathbf{\Gamma}}_1^{(0)}, \hat{\sigma}_1^{(0)}), \dots, (\hat{\mathbf{B}}_N^{(0)}, \hat{\mathbf{\Gamma}}_N^{(0)}, \hat{\sigma}_N^{(0)})$ obtained as follows:

- (a) Choose a random subsample of $\max(p_1, \dots, p_m)$ integers from the first n integers.
- (b) Calculate OLS on the corresponding rows of \mathbf{Y} and $\tilde{\mathbf{X}}$, giving $\hat{\mathbf{B}}_l^{(0)}$. If this subset yields a singular solution in a block, randomly increase the number of observations in that block, until a nonsingular OLS solution is obtained.
- (c) Compute the robust covariance matrix of the residuals by their one-step M-estimator [26, pag. 197]:
 - Set the initial covariance matrix $\hat{\Sigma}^{(0)} = \text{diag}(\text{mad}(\mathbf{Y} - \tilde{\mathbf{X}}\hat{\mathbf{B}}_l^{(0)}))^2$ where the median absolute deviation (mad) is computed on each column of the residual matrix.
 - Compute the robust distances $d_i = \sqrt{e_i(\hat{\mathbf{B}}_l^{(0)})'(\hat{\Sigma}^{(0)})^{-1}e_i(\hat{\mathbf{B}}_l^{(0)})}$.
 - Update the initial covariance matrix: $\hat{\Sigma}_l^{(0)} = \frac{1}{n} \sum_{i=1}^n w(d_i) e_i(\hat{\mathbf{B}}_l^{(0)}) e_i(\hat{\mathbf{B}}_l^{(0)})'$.

Then we set $\hat{\mathbf{\Gamma}}_l^{(0)} = |\hat{\Sigma}_l^{(0)}|^{-1/m} \hat{\Sigma}_l^{(0)}$ and $\hat{\sigma}_l^{(0)} = \text{med}_{i=1}^n \sqrt{e_i(\hat{\mathbf{B}}_l^{(0)})'(\hat{\mathbf{\Gamma}}_l^{(0)})^{-1}e_i(\hat{\mathbf{B}}_l^{(0)})}$ for all $l = 1, \dots, N$. Next, those estimates are refined by performing k so-called I -steps, resulting in

$$(\hat{\mathbf{B}}_1^{(k)}, \hat{\mathbf{\Gamma}}_1^{(k)}, \hat{\sigma}_1^{(k)}), \dots, (\hat{\mathbf{B}}_N^{(k)}, \hat{\mathbf{\Gamma}}_N^{(k)}, \hat{\sigma}_N^{(k)}).$$

The j th I -step to refine the estimate $(\hat{\mathbf{B}}_l^{(j-1)}, \hat{\mathbf{\Gamma}}_l^{(j-1)}, \hat{\sigma}_l^{(j-1)})$ goes as follows:

1. Refine the scale: $\hat{\sigma}_l^{(j)} = \hat{\sigma}_l^{(j-1)} \sqrt{\frac{1}{nb} \sum_{i=1}^n \rho \left(\frac{\sqrt{e_i(\hat{\mathbf{B}}_l^{(j-1)})'(\hat{\mathbf{\Gamma}}_l^{(j-1)})^{-1}e_i(\hat{\mathbf{B}}_l^{(j-1)})}}{\hat{\sigma}_l^{(j-1)}} \right)}$.
2. Use $\hat{\sigma}_l^{(j)}$ to compute weights $w_i^{(j)} = \frac{\rho'(u_i)}{u_i}$ with $u_i = \frac{\sqrt{e_i(\hat{\mathbf{B}}_l^{(j-1)})'(\hat{\mathbf{\Gamma}}_l^{(j-1)})^{-1}e_i(\hat{\mathbf{B}}_l^{(j-1)})}}{\hat{\sigma}_l^{(j)}}$.
3. Update $\hat{\mathbf{B}}_l^{(j-1)}$ following equation (6): Let $\mathbf{D} = \text{diag}(w_i^{(j)})$ and $\mathbf{W} = (\sigma_l^{(j)})^{-2}(\hat{\mathbf{\Gamma}}_l^{(j-1)})^{-1} \otimes \mathbf{D}$, then $\hat{\beta}^{(j)} = (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathbf{y}$ and $\hat{\mathbf{B}}_l^{(j)} = \text{diag}(\hat{\beta}_1^{(j)}, \dots, \hat{\beta}_m^{(j)})$.
4. Update $\hat{\Sigma}$ following equation (7): $\hat{\Sigma}_l^{(j)} = m(\mathbf{Y} - \tilde{\mathbf{X}}\hat{\mathbf{B}}_l^{(j)})'\mathbf{D}(\mathbf{Y} - \tilde{\mathbf{X}}\hat{\mathbf{B}}_l^{(j)}) / \sum_{i=1}^n v(u_i)$, which leads to the refinement $\hat{\mathbf{\Gamma}}_l^{(j)} = |\hat{\Sigma}_l^{(j)}|^{-1/m} \hat{\Sigma}_l^{(j)}$.

After performing k I -steps, the scale $\hat{\sigma}_l^{(k)}$ is improved for each $(\hat{\mathbf{B}}_l^{(k)}, \hat{\mathbf{\Gamma}}_l^{(k)}, \hat{\sigma}_l^{(k)})$ by iteratively solving

$$\hat{\sigma}_l^{(k+1)} = \hat{\sigma}_l^{(k)} \sqrt{\frac{1}{nb} \sum_{i=1}^n \rho \left(\frac{\sqrt{e_i(\hat{\mathbf{B}}_l^{(k)})'(\hat{\mathbf{\Gamma}}_l^{(k)})^{-1}e_i(\hat{\mathbf{B}}_l^{(k)})}}{\hat{\sigma}_l^{(k)}} \right)} \quad (8)$$

until convergence while keeping $\hat{\mathbf{B}}_l^{(k)}$ and $\hat{\mathbf{\Gamma}}_l^{(k)}$ fixed. We keep the v refined estimates $(\hat{\mathbf{B}}_1^{(*)}, \hat{\mathbf{\Gamma}}_1^{(*)}, \hat{\sigma}_1^{(*)}), \dots, (\hat{\mathbf{B}}_v^{(*)}, \hat{\mathbf{\Gamma}}_v^{(*)}, \hat{\sigma}_v^{(*)})$ with the smallest fully iterated scales. Note that not all scales $\hat{\sigma}_l^{(k)}$, $l = 1, \dots, N$ need to be computed by solving (8). The first v scales $\hat{\sigma}_l^{(k)}$, $l = 1, \dots, v$ are always computed, but for $l > v$ the l th scale is only computed if

$$\frac{1}{n} \sum_{i=1}^n \rho \left(\frac{\sqrt{e_i(\hat{\mathbf{B}}_l^{(k)})'(\hat{\mathbf{\Gamma}}_l^{(k)})^{-1}e_i(\hat{\mathbf{B}}_l^{(k)})}}{A} \right) < b$$

where A is the maximum of the v best scales that were fully iterated so far. The v estimates $(\hat{\mathbf{B}}_1^{(*)}, \hat{\mathbf{\Gamma}}_1^{(*)}, \hat{\sigma}_1^{(*)}), \dots, (\hat{\mathbf{B}}_v^{(*)}, \hat{\mathbf{\Gamma}}_v^{(*)}, \hat{\sigma}_v^{(*)})$ with the smallest scales need to be refined until convergence using I -steps as described above, and the final estimate $(\hat{\mathbf{B}}^{(F)}, \hat{\mathbf{\Gamma}}^{(F)}, \hat{\sigma}^{(F)})$ is the one with the smallest scale after full refinement. The final robust SUR estimates are then $\hat{\mathbf{B}} = \hat{\mathbf{B}}^{(F)}$ (or equivalently $\hat{\beta} = (\hat{\beta}'_1, \dots, \hat{\beta}'_m)'$) and $\hat{\Sigma} = (\hat{\sigma}^{(F)})^2 \hat{\mathbf{\Gamma}}^{(F)}$.

Note that the number of subsets N , the number of I -steps k and the number of refined estimates v can be chosen by the user. In our experience the settings $N = 500, k = 2$ and $v = 5$ (as in [23]) work well for many data sizes and contamination patterns, but using larger values for these parameters might be useful at large data sets with potentially a high contamination level.

4 Simulation study

In this section we study the performance of our FastSUR algorithm on artificial data sets. As a benchmark, we always compare our results with the FGLS algorithm, as computed within the R package `systemfit` [27].

We carried out an extensive simulation study on data sets of different dimension and here we report the results of two settings, namely, $A : n = 100, p_j = 5, m = 8$ and $B : n = 30, p_j = 3, m = 4$. For each simulation setting, we generated $K = 100$ data sets, with fixed β and fixed Σ . Each block contains the same number of explanatory variables p_j . The values of β_j are randomly drawn from a uniform distribution $U([0, 10])$ (although we could equally well fix them to zero due to the regression equivariance of the

estimators). For generating the covariance matrix Σ we followed the methodology of [28], taking $[1, 4]$ as the range for the variances. For each data set, the independent variables were generated from a $(p_j - 1)$ -variate standard normal distribution $N_{p_j-1}(0, \mathbf{I}_{p_j-1})$. We considered two different distributions for the error terms $\varepsilon = (\varepsilon'_1, \varepsilon'_2, \dots, \varepsilon'_m)'$. First, we studied normal errors generated from $N_{nm}(0, \Sigma \otimes \mathbf{I}_n)$ and secondly we considered heavy tailed errors following a Student distribution with 3 degrees of freedom, $t_3(0, \Sigma \otimes \mathbf{I}_n)$. The response variables were then computed according to model (1).

In order to contaminate the data, we replaced the first 5%, 10% and 30% of observations in each block by bad leverage points, by replacing some predictor variables with a random value from $U([20, 30])$ while keeping the response value unchanged. It is well-known that these type of outliers, being both outlying in the space of the predictor variables as in the errors, are considered the most influential type of outliers that often cause the regression estimates to be highly biased [29]. For each simulation setting and each data set, we applied both FGLS and robust FastSUR. We always used a breakdown value of 25%, except at those data sets with 30% contamination where the breakdown value was set to 50%.

In order to measure the performance of the estimators, we evaluated their bias and mean squared error:

$$Bias = \left\| \frac{1}{K} \sum_{k=1}^K \hat{\beta}^{(k)} - \beta \right\|$$

$$MSE = \frac{1}{K} \sum_{k=1}^K \left\| \hat{\beta}^{(k)} - \beta \right\|^2$$

where β is the true regression vector, and $\hat{\beta}^{(k)}$ is the FGLS or robust SUR estimate on the k th sample.

The results are given in Table 1 for data setting A and in Table 2 for setting B. It can be observed that for uncontaminated data sets, the FastSUR results are close to FGLS. When there is contamination, the bias and MSE of FGLS explode, whereas the robust FastSUR algorithm yields very satisfactory results that do not deviate much from the uncontaminated case. Only with a very large amount of outliers in each block, FastSUR has a slightly increased bias and MSE.

We also ran more simulations in which the outliers were differently positioned in each block, but the outcomes were always comparable to the results of Table 1 and 2. Whereas FGLS collapses at data sets with outliers, FastSUR is more resistant to them.

Table 1: Simulation results for data setting A with 0%, 5%, 10% and 30% contamination, and a normal or heavy-tailed error distribution.

<i>Outliers</i>	0%		5%		10%		30%	
	FGLS	FastSUR	FGLS	FastSUR	FGLS	FastSUR	FGLS	FastSUR
<i>Normal</i>								
Bias	0.212	0.212	15.203	0.241	23.084	0.223	33.599	0.259
MSE	3.145	3.176	358.724	3.565	884.582	3.529	1193.012	5.788
<i>t₃</i>								
Bias	0.305	0.301	16.537	0.292	24.247	0.313	32.463	0.782
MSE	9.405	9.545	359.924	6.918	906.993	7.577	1016.514	10.283

Table 2: Simulation results for data setting B with 0%, 5%, 10% and 30% contamination, and a normal or heavy-tailed error distribution.

<i>Outliers</i>	0%		5%		10%		30%	
	FGLS	FastSUR	FGLS	FastSUR	FGLS	FastSUR	FGLS	FastSUR
<i>Normal</i>								
Bias	0.213	0.204	11.076	0.170	11.433	0.179	16.104	0.232
MSE	3.442	3.591	143.736	3.305	154.008	3.423	288.558	8.167
<i>t₃</i>								
Bias	0.376	0.393	11.084	0.311	11.415	0.348	16.116	2.176
MSE	8.642	8.744	144.803	7.669	156.985	8.198	274.167	10.892

5 Outlier detection

Applying FastSUR on a real data set does not only result in robust parameter estimates, it also provides diagnostics for outlier detection. First, based on the multiple regression model (2), we can quantify the outlyingness of observations within one block by their *standardized residual*. For the i th observation of block j , it is equal to r_{ij} being the i th value of the standardized residual vector $r_j = (y_j - \mathbf{X}_j \hat{\beta}_j) / \hat{\sigma}_{jj} = (r_{1j}, \dots, r_{nj})'$. Under gaussian errors, an observation is typically considered to be outlying if its absolute standardized residual exceeds 2.5.

Next, we can consider the multivariate regression model (3). The i th row in the data

matrices \mathbf{X} and \mathbf{Y} then corresponds with the measurements of the i th observation in each block ($i = 1, \dots, n$). Consequently, the i th *residual distance*

$$\text{ResD}_i = \sqrt{e_i(\hat{\mathbf{B}})' \hat{\mathbf{\Sigma}}^{-1} e_i(\hat{\mathbf{B}})} \quad (9)$$

can be used to detect outlying behavior of one of the n rows. Under normal errors, a residual distance larger than $\sqrt{\chi_{m,0.975}^2}$ (the square root of the 0.975 quantile of the χ_m^2 distribution) is flagged as being unusually large.

To illustrate these diagnostics, we study the Foreign Direct Investment (FDI) of six countries (India, Indonesia, Columbia, Mexico, Turkey and Chile) over the period 1981-2012. The countries constitute the $m = 6$ blocks in our SUR model, with measurements of $n = 32$ years per country. FDI is considered to be the main source of economic growth. In [30] it is referred that FDI is mostly defined as capital flows resulting from the behavior of multinational companies (MNCs) and the factors to affect the behavior of MNCs may also affect the magnitude and the direction of FDI. As predictor variables we include several macroeconomic variables: the growth rate per capita GPD, the rate of inflation measured by annual percentage change of consumer prices, and the degree of openness which is computed as the sum of nominal export and import divided by the nominal GDP. The data are collected from The World Bank, World Development Indicators ¹.

The relationship between FDI and its potential determinant variables has been a prominent topic in the last two decades. In several studies SUR and panel models were applied [31]. The reason for using the SUR model is the possible existence of common factors that influence all the countries at the same time and bring about the cross correlations between the error terms.

We apply both FLGS and FastSUR (with 50% breakdown value) on our data set and compare the residuals by means of several outlier maps [32]. First, for each country j we plot the FGLS standardized residuals r_{ij} versus the Mahalanobis distance of the observed predictor variables MD_{ij} (for $i = 1, \dots, n$). The latter are computed as

$$\text{MD}_{ij} = \sqrt{(x_{ij} - \bar{x}_j)' \mathbf{S}_j^{-1} (x_{ij} - \bar{x}_j)} \quad (10)$$

with \bar{x}_j and \mathbf{S}_j the mean and covariance matrix of $\mathbf{X}_j = (x_{1j}, \dots, x_{nj})'$. If the predictor variables are normally distributed, the squared Mahalanobis distances approximately have a $\chi_{p_j}^2$ distribution, hence we set the cutoff to $\sqrt{\chi_{p_j,0.975}^2}$. For Indonesia this yields the residual plot in Figure 1(a). The lines indicate the cutoff values for the residuals and the Mahalanobis distances. Observations exceeding these cutoff values are considered to

¹website: <http://data.worldbank.org/data-catalog/world-development-indicators>

be outliers according to the FGLS estimates. We see that for the years 1997, 2000 and 2012 the standardized residuals are slightly larger than expected, but the macroeconomic predictor variables do not have abnormal values. These type of observations are called vertical outliers. In 1998 the FDI can be well predicted, but the larger MD indicates that the explanatory variables deviate from the other years. This observation is called a good leverage point. Looking at the raw data for Indonesia we notice that all predictor variables have an extreme value in 1998 (a negative growth rate of -14%, a huge inflation rate of 75% and a degree of openness which is over 98%). Figure 1(b) shows the FastSUR results. Here the vertical axis shows the standardized residuals based on the FastSUR estimates, whereas the horizontal axis depicts the robust distances

$$RD_{ij} = \sqrt{(x_{ij} - \hat{\mu}_j)' \hat{\Sigma}_j^{-1} (x_{ij} - \hat{\mu}_j)}$$

with $\hat{\mu}_j$ and $\hat{\Sigma}_j$ the FastS estimates of the center and scatter of \mathbf{X}_j . Now, the years 2001 and 2003 show up as very clear vertical outliers, and also in 2000 the fit is not very good. In all these three years the FDI was negative. Furthermore, 1998 is indicated as a very prominent good leverage point. Note that when we analyze solely the data from Indonesia by means of OLS and robust S-regression, we obtain the residual plots of Figure 1(c) and (d). In both cases the year 2003 is not detected as a vertical outlier.

The residual distance plot, showing all n residual distances (9), is depicted in Figure 2(a) for FGLS and (b) for robust SUR. Here, we notice a huge difference between both estimates. Whereas the SUR estimates flag none of the years as outlying (although there is an increasing trend), robust SUR finds that most of the recent years have a very different behavior. This corresponds with the global financial and economic crisis period which has a substantial effect on the emerging countries.

6 Actuarial application

Stochastic claims reserving is a major actuarial problem in general insurance and with the introduction of new regulatory guidelines for the insurance business there is a growing awareness that modern statistical techniques should be used. Claims reserves are often the largest position on the liability side of the balance sheet of a general insurance company. We study claims that take months or years to emerge depending on the complexity of the damage. The delay in payment is, for example, due to long legal procedures or difficulties in determining the size of the claims. Therefore, insurers have to build up reserves enabling them to pay the outstanding claims and to meet claims arising in the

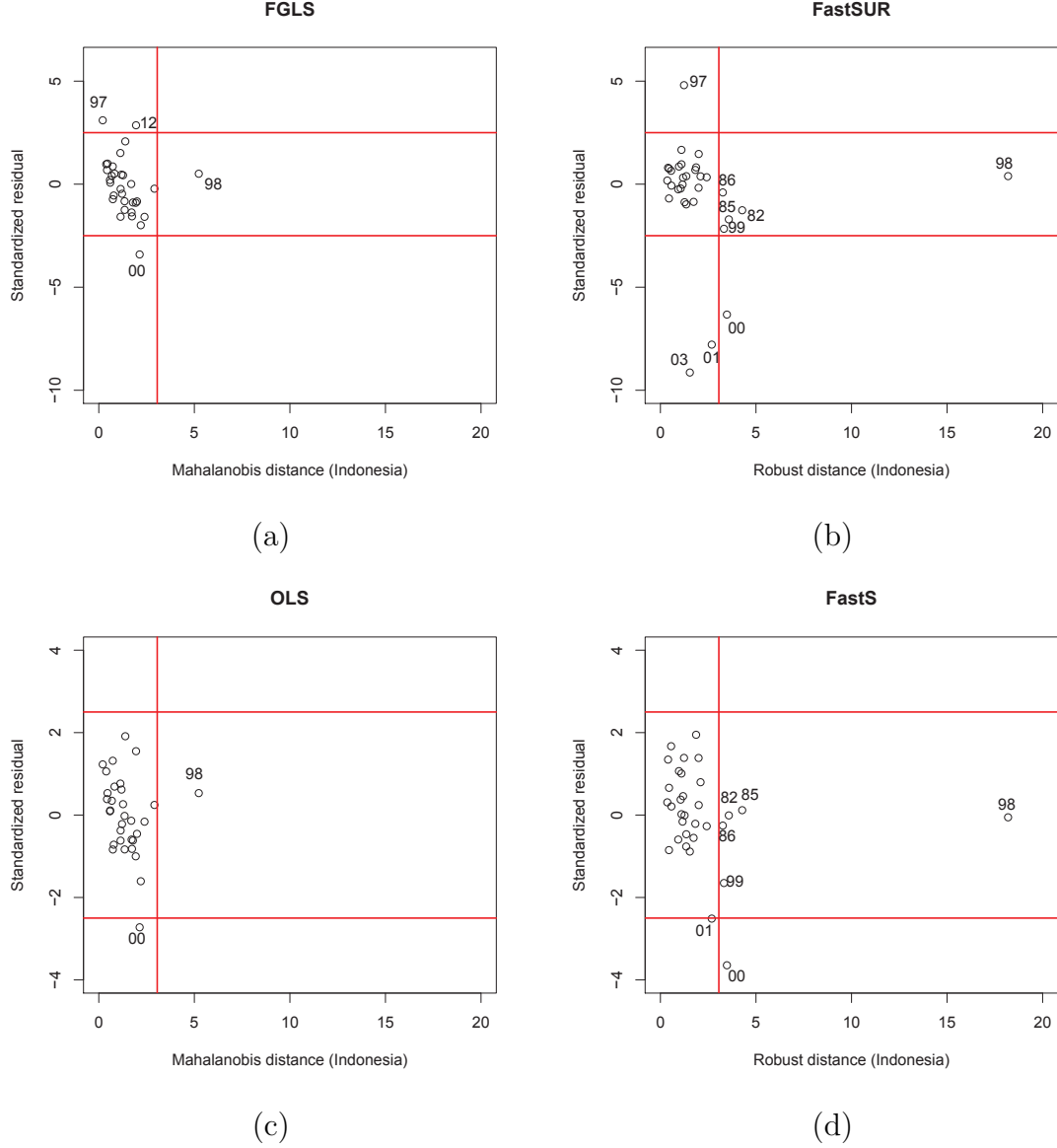


Figure 1: Regression diagnostic plots of Indonesia based on (a) FGLS; (b) robust SUR; (c) OLS and (d) robust S-regression.

future on the written contracts. In this section, we describe how the reserve estimates can be obtained using the SUR technique. Our FastSUR algorithm will then be plugged in in this methodology and its good performance will be illustrated on a real data set.

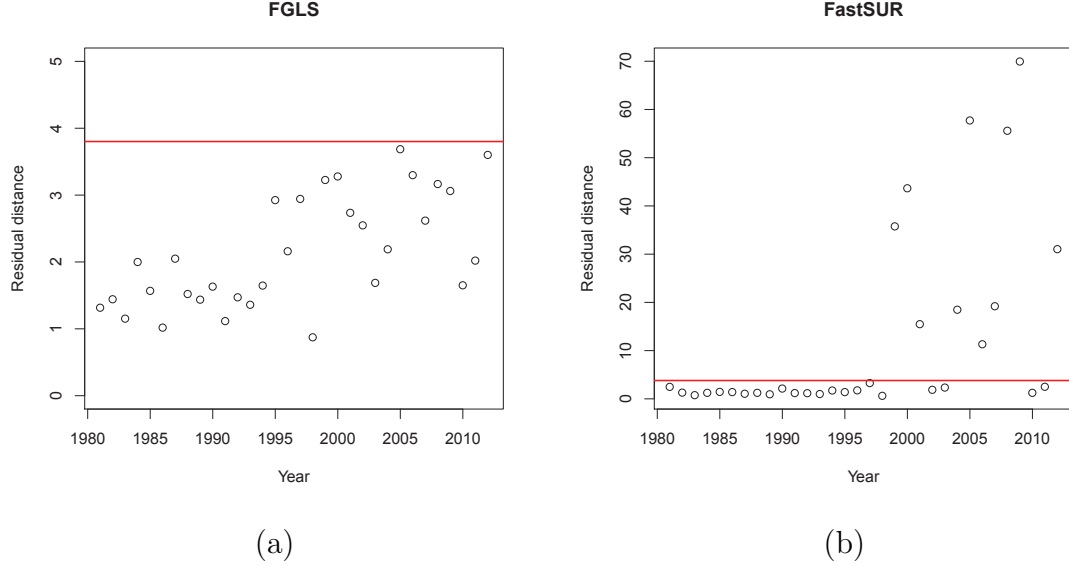


Figure 2: Residual distance plot of the FDI data set based on (a) classical estimates; (b) robust estimates.

6.1 Chain Ladder method

We assume that C_{ik} (for $1 \leq i \leq I$ and $1 \leq k \leq K$) are the cumulative claims amount of accident year i and development year k . For representation of the data it is common to use a run-off triangle as in Table 3.

Table 3: Run-off triangle.							
<u>development year</u> <u>accident year</u>	1	2	...	k	...	$K-1$	K
1	$C_{1,1}$	$C_{1,2}$...	$C_{1,k}$...	$C_{1,K-1}$	$C_{1,K}$
2	$C_{2,1}$	$C_{2,2}$...	$C_{2,k}$...	$C_{2,K-1}$	
...		
i	$C_{i,1}$	$C_{i,2}$...	$C_{i,k}$			
...				
I	$C_{I,1}$						

The ultimate goal of claims reserving boils down to completing the triangle into a square (or a rectangle if estimates are required pertaining to development years of which no data are recorded at hand) since the total of the values found in the lower right triangle equals the overall reserve R that will need to be paid in future. The Chain Ladder (CL) method is the most popular method for estimating R . Many problems related with the

CL method have already been solved in literature. For example, in [33] a tail factor is included, whereas [34] and [35] deal with the presence of negative values in the run-off triangle.

In practice, a general insurance company subdivides portfolios into several correlated subportfolios, such that each subportfolio (which is represented by a run-off triangle) satisfies certain homogeneity properties. However, the CL method applies to a single run-off triangle and therefore neglects the contemporaneous correlations existing between subportfolios. Since it is well-known that the CL predictions for the sum of several run-off triangles in general differ from the sum of the Chain Ladder predictions for the single run-off triangles [36], the claims reserving problem is recently studied in a multivariate context [37, 38]. In [39] and [40] a Multivariate Chain Ladder (MCL) model is introduced, where the multivariate estimators take into account the dependence structure between the subportfolios and which are optimal in terms of a classical optimality criterion. In [41] a conditional mean squared error of prediction (MSEP) estimator for this multivariate version is provided, which is useful to quantify the uncertainties in the reserve estimates.

6.2 MCL in SUR framework

In [14] it is recently shown that the estimators in the MCL model can find their equivalents in the SUR framework. We give a brief description of this General Multivariate Chain Ladder (GMCL) model and refer to [14] for more details. Assume that we have m correlated run-off triangles with I accident and K development years (for simplicity, we assume $I = K$) and that the claims from different accident years are independent. Denote $\tilde{C}_{i,k} = \left(C_{i,k}^{(1)}, \dots, C_{i,k}^{(m)} \right)'$ as the vector of cumulative claims at accident year i and development year k and consider the following model structure for development period k (i.e. from development year k to $k + 1$):

$$\tilde{C}_{i,k+1} = \mathcal{B}_k \tilde{C}_{i,k} + \varepsilon_{i,k} \quad \text{for } i = 1, \dots, I - k. \quad (11)$$

Here \mathcal{B}_k is the corresponding $m \times m$ development matrix that contains the development parameters $\beta_j = (\beta_{j1}, \dots, \beta_{jm})'$ for run-off triangle $j \leq m$ in the j th row and $\varepsilon_{i,k}$ are symmetrically distributed errors. Therefore, the development of one run-off triangle in development period k can depend on the claims in the other run-off triangles at development year k . Moreover, it is assumed that

$$E(\varepsilon_{i,k} | \mathcal{D}_{i,k}) = 0 \quad (12)$$

$$\text{Cov}(\varepsilon_{i,k} | \mathcal{D}_{i,k}) = \text{diag}(\tilde{C}_{i,k})^{1/2} \Sigma_k \text{diag}(\tilde{C}_{i,k})^{1/2} \quad (13)$$

where $\mathcal{D}_{i,k} = \{C_{i,k}^{(j)} | i \leq k, j = 1, \dots, m\}$ is the set of claims for accident year i up to and including development year k and Σ_k is a symmetric positive definite $m \times m$ matrix.

In [14] the model structure for development period k (11) has been rewritten as the following system of equations

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \dots & 0 \\ 0 & \mathbf{X}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{X}_m \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_m \end{pmatrix} \quad (14)$$

where for $j = 1, 2, \dots, m$ and $n = I - k$ it holds that

- $y_j = C_{\leq, k+1}^{(j)}$ is the $n \times 1$ vector of all observed losses at development year $k+1$ from the j th triangle
- $\mathbf{X}_j = (C_{<, k}^{(1)}, \dots, C_{<, k}^{(m)})$ is the $n \times m$ matrix of the first n observations at development year k from each triangle (hence $\mathbf{X}_1 = \dots = \mathbf{X}_m$)
- ε_j is the $n \times 1$ vector of error terms in the j th equation.

From (12) and (13) it follows that

$$\text{Cov}(\varepsilon) = E(\varepsilon\varepsilon') = \text{diag}(V)^{1/2}(\Sigma_k \otimes \mathbf{I}_n)\text{diag}(V)^{1/2}$$

where $\varepsilon = (\varepsilon'_1, \dots, \varepsilon'_m)'$ and $V = ((C_{<, k}^{(1)})', \dots, (C_{<, k}^{(m)})')'$ is the $nm \times 1$ vector of the first n observed claims at development year k . Pre-multiplying both sides of model (14) by $\text{diag}(V)^{-1/2}$ leads to a regression model whose error covariance matrix $\text{Cov}(\varepsilon^*)$ is consistent with the SUR assumption:

$$\text{Cov}(\varepsilon^*) = \text{diag}(V)^{-1/2} \text{Cov}(\varepsilon) \text{diag}(V)^{-1/2} = \Sigma_k \otimes \mathbf{I}_n.$$

After estimating the development parameters β_j ($j = 1, \dots, m$) using the FGLS estimation procedure consecutively for all development periods $k = 1, \dots, K - 1$, the overall reserve estimate \hat{R} (for m triangles simultaneously) can be obtained.

In the univariate setting ($m = 1$) the sensitivity of the CL method to outliers has already been demonstrated [42, 43]. Even one outlier can lead to a huge over- or underestimation of the overall reserve estimate. Since the traditional SUR estimator is not robust, the corresponding GMCL estimates are also unreliable in the presence of outliers. Note that robustness now even plays a more important role, since an outlier in one of the run-off triangles may now also affect the estimates of outstanding claim amounts of the other run-off triangles. When we plug in our robust SUR algorithm in the GMCL model, we obtain robust reserve estimates and diagnostics for outlier detection.

6.3 Real example

To study the performance of the robust GMCL estimator, we focus on the real example that is presented in [14]. The studied portfolio from Schedule P of General Accident Insurance Company (published by NAIC) consists of three run-off triangles that are given in Table 4, 5 and 6.

Table 4: Cumulative paid triangle from Personal Auto.

	1	2	3	4	5	6	7	8	9	10
1	101125	209921	266618	305107	327850	340669	348430	351193	353353	353584
2	102541	203213	260677	303182	328932	340948	347333	349813	350523	
3	114932	227704	298120	345542	367760	377999	383611	385224		
4	114452	227761	301072	340669	359979	369248	373325			
5	115597	243611	315215	354490	372376	382738				
6	127760	259416	326975	365780	386725					
7	135616	262294	327086	367357						
8	127177	244249	317972							
9	128631	246803								
10	126288									

Table 5: Cumulative incurred triangle from Personal Auto.

	1	2	3	4	5	6	7	8	9	10
1	325423	336426	346061	347726	350995	353598	354797	355025	354986	355363
2	323627	339267	344507	349295	351038	351583	352050	352231	352193	
3	358410	386330	385684	384699	387678	387954	388540	389436		
4	405319	396641	391833	384819	380914	380163	379706			
5	434065	429311	422181	409322	394154	392802				
6	417178	422307	413486	406711	406503					
7	398929	398787	398020	400540						
8	378754	361097	369328							
9	351081	335507								
10	329236									

Similar as in [14], we fit the multivariate GMCL model for development years 1-7 and the univariate CL model for development years 8-10 (since the gain of increasing model complexity after year 7 is minor). Applying the classical and the robust methods on the original data yields respectively an overall reserve estimate of 1 049 664 and 1 052 546,

Table 6: Cumulative paid triangle from Commercial Auto.

	1	2	3	4	5	6	7	8	9	10
1	19827	44449	61205	77398	88079	95695	99853	104789	105427	106690
2	22331	48480	68789	92356	104958	112399	115638	117415	118571	
3	22533	44484	65691	88435	102044	112672	115973	118359		
4	23128	51328	81542	98063	113149	121515	124347			
5	25053	57220	84607	104936	117663	126180				
6	30136	64767	92288	108835	121326					
7	34764	69125	91354	111987						
8	31803	63471	92439							
9	40559	77667								
10	46285									

which is very close to each other. When we multiply for example claim $C_{2,2}^{(1)}$ by 10, then the classical and robust overall reserve estimate equals respectively 825 530 and 1 048 768. When there is a significant difference between the classical and robust overall reserve estimate, then it is highly recommended to examine the data more closely and to study the most influential observations. When plotting the residual distances obtained with the classical and robust GMCL, we see from Figure 3 that only the robust version succeeds in detecting the outlier. Let us now multiply $C_{2,2}^{(1)}$ and $C_{2,2}^{(2)}$ by 1.2 and divide $C_{2,2}^{(3)}$ by 1.2. The

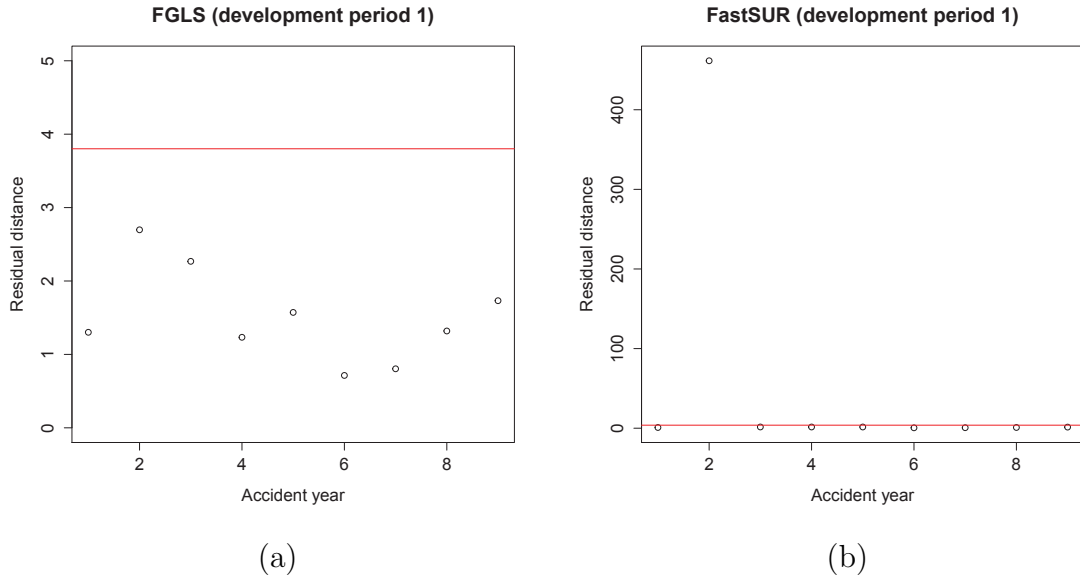


Figure 3: Residual distance plot for the first contamination based on (a) classical GMCL estimates; (b) robust GMCL estimates.

overall reserve estimates for the classical and robust GMCL method equals respectively 1 036 407 and 1 048 768. Although this (multivariate) outlier configuration does not have a very large impact on the overall reserve estimates, the robust GMCL method still succeeds in detecting the atypical observation as we can see in Figure 4. Also here the non-robust GMCL does not indicate any anomaly.

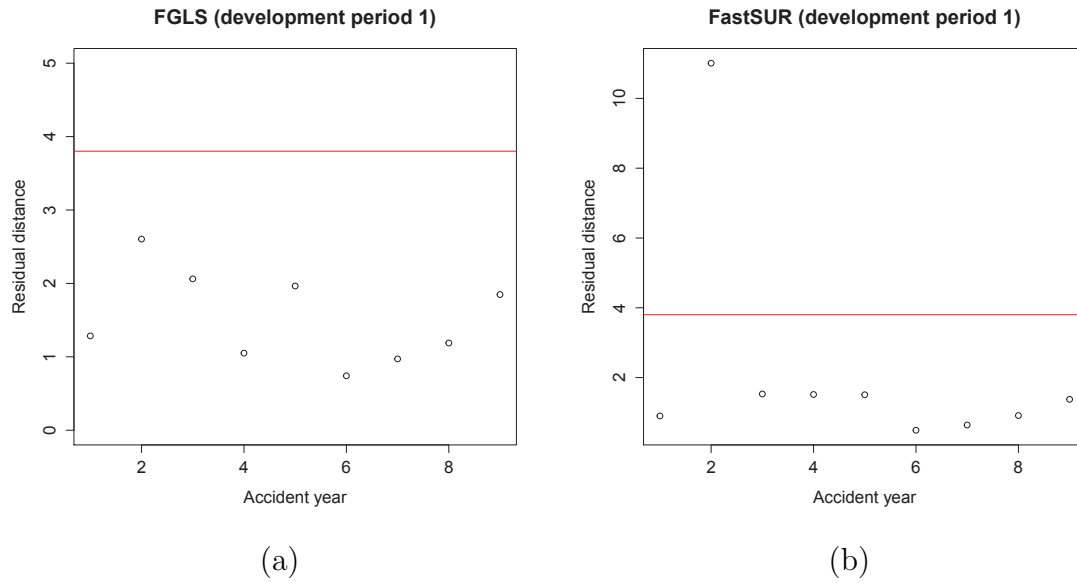


Figure 4: Residual distance plot for the second contamination based on (a) classical GMCL estimates; (b) robust GMCL estimates.

7 Conclusion and outlook

Our new algorithm for robust SUR provides robust parameter estimates and useful outlier diagnostics, as illustrated on two real data sets. Several extensions are interesting to study. Robust inference might be obtained using the robust bootstrap as in [20]. This would provide standard errors of the robust point estimates, and hence a fast approach towards robust model selection. High collinearity among the predictor variables could be alleviated by applying a robust PCA method to the predictor variables as in [44] or by generalizing the robust PLS method in [45]. When the set of predictor variables includes categorical ones, the algorithm might need many more random subsets than the current $N = 500$. Techniques as in [46] or [47] could be extended towards this setting. Also the use of multivariate τ -estimators could be considered, by combining ideas from [48] and [49].

Applied to the framework of stochastic loss reserving for general insurance we obtain robust estimates in the GMCL model. The proposed methodology is helpful to build up a more realistic reserve, certainly when used in addition to the classical GMCL estimates. Another advantage of the SUR model in the claims reserving context is that it also allows structural connections among triangles and hence allows to construct flexible models. In the classical approach, this is already described and illustrated in [14]. Studying this using the robust GMCL model could also be an interesting topic for future research.

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